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Green function estimates for relativistic stable processes in half-space-like open sets

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Abstract

In this paper, we establish sharp two-sided estimates for the Green functions of relativistic stable processes (i.e. Green functions for non-local operators $m-(m^{2/\alpha}-\Delta)^{\alpha/2}$) in half-space-like $C^{1,1}$ open sets. The estimates are uniform in $m \in (0, M]$ for each fixed $M \in (0, \infty)$. When $m \downarrow 0$, our estimates reduce to the sharp Green function estimates for $-(-\Delta)^{\alpha/2}$ in such kind of open sets that were obtained recently in Chen and Tokle [12]. As a tool for proving our Green function estimates, we show that a boundary Harnack principle for X^m, which is uniform for all $m \in (0, \infty)$, holds for a large class of non-smooth open sets.

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1. Introduction

Suppose that X is a Markov process in $\mathbb{R}^d$ and D is an open subset of $\mathbb{R}^d$. The Green function $G_D(x, y)$ of X in D is the occupation density of the subprocess X^D of X killed upon exiting D and is a very important quantity in probability theory. It is also very important in PDE since it can be used to solve the Poisson equations associated with the generator of X in D with zero

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exterior condition. In general, one cannot get a closed form analytic expression of $G_D(x, y)$, so two-sided sharp estimates on $G_D(x, y)$ are very valuable. In this paper, we will derive sharp estimates on the Green functions for relativistic α-stable processes in half-space-like $C^{1,1}$ open sets D (or, equivalently, for the non-local operators $m - (m^{2/\alpha} - \Delta)^{\alpha/2}$ in D with zero exterior condition).

Let $d \geq 1$ and $\alpha \in (0, 2)$. Recall that for any $m > 0$, a relativistic α-stable process X^m in $\mathbb{R}^d$ with mass m is a Lévy process with characteristic function given by

$$\mathbb{E}\left[\exp\left(i\xi \cdot (X_t^m - X_0^m)\right)\right] = \exp\left(-t\left(\left(|\xi|^2 + m^{2/\alpha}\right)^{\alpha/2} - m\right)\right), \quad \xi \in \mathbb{R}^d. \tag{1.1}$$

The limiting case X^0, corresponding to $m = 0$, is a (rotationally) symmetric α-stable (Lévy) process in $\mathbb{R}^d$ which we will simply denote as X. The infinitesimal generator of X^m is $m-(m^{2/\alpha}-\Delta)^{\alpha/2}$. Note that when $m = 1$, this infinitesimal generator reduces to $1-(1-\Delta)^{\alpha/2}$. Thus the 1-resolvent kernel of the relativistic α-stable process X^1 in $\mathbb{R}^d$ is just the Bessel potential kernel. When $\alpha = 1$, the infinitesimal generator reduces to the so-called free relativistic Hamiltonian $m - \sqrt{-\Delta + m^2}$. The operator $m - \sqrt{-\Delta + m^2}$ is important in mathematical physics due to its correspondence with the kinetic energy of a relativistic particle with mass m, see [18]. Physical models related to this operator have been much studied over the past 30 years and there exists a huge literature on the properties of relativistic Hamiltonians (see, for example, [6,14,18,24,25,32]). For recent papers in the mathematical physics literature related to the relativistic Hamiltonian, we refer the readers to [13,15,16,30] and the references therein. Various fine properties of relativistic α-stable processes have been studied recently in [4,10,11,17,19,20,23,26]. In particular, the following sharp estimates described in Theorem 1.1 below on the transition densities $p_D^m(t, x, y)$ of X^m in $C^{1,1}$ open sets D have been obtained very recently in [8]. Recall that an open set D in $\mathbb{R}^d$ (when $d \geq 2$) is said to be a $C^{1,1}$ open set if there exist a localization radius $R > 0$ and a constant $\Lambda_0 > 0$ such that for every $z \in \partial D$, there exist a $C^{1,1}$-function $\phi = \phi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = 0, \nabla\phi(0) = (0, \ldots, 0)$, $\|\nabla\phi\|_\infty \leq \Lambda_0$, $|\nabla\phi(x) - \nabla\phi(z)| \leq \Lambda_0|x - z|$, and an orthonormal coordinate system $CS_z y = (y_1, \ldots, y_{d-1}, y_d) := (\widehat{y}, y_d)$ with origin at z such that $B(z, R) \cap D = \{y = (\widehat{y}, y_d) \in B(0, R) \text{ in } CS_z : y_d > \phi(\widehat{y})\}$. We call (R, Λ_0) the $C^{1,1}$ characteristics of D. By a $C^{1,1}$ open set in $\mathbb{R}$ we mean an open set which can be expressed as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is positive. Note that a $C^{1,1}$ open set may be unbounded and disconnected. In this paper, we use ":=" as a way of definition. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

Theorem 1.1. *Suppose that D is a $C^{1,1}$ open set in $\mathbb{R}^d$ with $C^{1,1}$ characteristics (R, Λ_0). Let $\delta_D(x)$ be the distance between x and D^c.*

(i) *For any $M > 0$ and $T > 0$, there exist $C_k = C_k(d, \alpha, R, \Lambda_0, M, T) > 1$, $k = 1, 2$, such that for any $m \in (0, M]$ and $(t, x, y) \in (0, T] \times D \times D$,*

$$\frac{1}{C_1}\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t\phi(C_2 m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\right) \leq p_D^m(t, x, y)$$
$$\leq C_1\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t\phi(m^{1/\alpha}|x-y|/C_2)}{|x-y|^{d+\alpha}}\right),$$

where $\phi(r) = e^{-r}(1 + r^{(d+\alpha-1)/2})$.

(ii) *Suppose in addition that D is bounded. For any $M > 0$ and $T > 0$, there exist $C_k = C_k(D, M, T) > 0$, $k = 3, 4$, such that for any $m \in (0, M]$ and $(t, x, y) \in [T, \infty) \times D \times D$,*

$$C_3 e^{-t\lambda_1^{\alpha,m,D}} \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} \le p_D^m(t, x, y) \le C_4 e^{-t\lambda_1^{\alpha,m,D}} \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2},$$

where $\lambda_1^{\alpha,m,D} > 0$ is the smallest eigenvalue of the restriction of $(m^{2/\alpha} - \Delta)^{\alpha/2} - m$ to D with zero exterior condition.

When D is a bounded $C^{1,1}$ open set, integrating the estimates on $p_D^m(t, x, y)$ from Theorem 1.1 with respect to t yields sharp two-sided estimates on the Green function $G_D^m(x, y) := \int_0^\infty p_D^m(t, x, y)dt$. To state this result, we define a function V_D^α on $D \times D$ by

$$V_D^\alpha(x, y) := \begin{cases} \left(1 \wedge \dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|^\alpha}\right)|x-y|^{\alpha-d} & \text{when } d > \alpha, \\ \log\left(1 + \dfrac{\delta_D(x)^{1/2}\delta_D(y)^{1/2}}{|x-y|}\right) & \text{when } d = 1 = \alpha, \\ \left(\delta_D(x)\delta_D(y)\right)^{(\alpha-1)/2} \wedge \dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|} & \text{when } d = 1 < \alpha. \end{cases}$$

The following result is given in [8, Theorem 1.2].

Theorem 1.2. *Let $M > 0$ be a constant and D a bounded $C^{1,1}$-open set in $\mathbb{R}^d$. Then there exist positive constants $C_5 < C_6$ depending only on D, α, M such that for all $m \in (0, M]$ and $(x, y) \in D \times D$,*

$$C_5 V_D^\alpha(x, y) \le G_D^m(x, y) \le C_6 V_D^\alpha(x, y).$$

The above theorem implies that, in any bounded $C^{1,1}$ open set D and for any $m \in (0, M]$, the Green function $G_D^m(x, y)$ is uniformly comparable to the Green function $G_D(x, y)$ of the stable process X in D. This comparability cannot be true when D is unbounded. The objective of this paper is to study sharp Green function estimates of X^m in a large class of unbounded $C^{1,1}$ open sets, i.e., the half-space-like $C^{1,1}$ open sets.

Following [12], we say an open set D in $\mathbb{R}^d$ is half-space-like if there is an orthonormal coordinate system $y = (y_1, \ldots, y_d)$ for $\mathbb{R}^d$ so that $H_a \subset D \subset H_b$ for some real numbers $a > b$. Here for $a \in \mathbb{R}$, $H_a := \{(y_1, \ldots, y_d) \in \mathbb{R}^d : y_d > a\}$. Although large time heat kernel estimates when D is unbounded are unavailable, by using the short time heat kernel estimates in Theorem 1.1(i), the uniform Harnack inequality (Theorem 2.3), the uniform boundary Harnack principle (Theorem 2.6), the two-sided Green function estimates on the upper half space from [17] (where some corrections and modifications are needed, see Theorem 3.1 below for details) and a comparison idea from [12], we are able to obtain sharp two-sided estimates on the Green function $G_D^m(x, y)$ when D is a half-space-like $C^{1,1}$ open set. A half-space-like $C^{1,1}$ open set in $\mathbb{R}$ is the disjoint union of finitely many bounded open intervals and an unbounded open interval. To state our result, we define a function $\widetilde{V}_D^{\alpha,m}$ on $D \times D$ as follows: for $m > 0$, let $\varphi_m(r) = r^{\alpha/2} + m^{(2-\alpha)/(2\alpha)}r$. When $d \ge 3$,

$$\widetilde{V}_D^{\alpha,m}(x,y) := \begin{cases} \left(m^{(2-\alpha)/\alpha} \wedge \dfrac{\varphi_m(\delta_D(x))\varphi_m(\delta_D(y))}{|x-y|^2}\right) |x-y|^{2-d} & \text{when } |x-y| > 3m^{-1/\alpha}, \\ \left(1 \wedge \dfrac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2} |x-y|^{\alpha-d} & \text{when } |x-y| \le 3m^{-1/\alpha}; \end{cases}$$

when $d = 2$,

$$\widetilde{V}_D^{\alpha,m}(x,y) := \begin{cases} \log\left(\left(1 + m^{(2-\alpha)/\alpha}\dfrac{\varphi_m(\delta_D(x))\varphi_m(\delta_D(y))}{|x-y|^2}\right)\right)^{m^{(2-\alpha)/\alpha}} \\ \quad \text{when } |x-y| > 3m^{-1/\alpha}, \\ \left(1 \wedge \dfrac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2} |x-y|^{\alpha-2} + m^{(2-\alpha)/\alpha}\log(1 \vee m^{1/\alpha}(\delta_D(x) \wedge \delta_D(y))) \\ \quad \text{when } |x-y| \le 3m^{-1/\alpha}; \end{cases}$$

when $d = 1 < \alpha$,

$$\widetilde{V}_D^{\alpha,m}(x,y) := \begin{cases} \dfrac{e^{-m^{1/\alpha}|x-y|}}{|x-y|^{1-(\alpha/2)}}(m^{-1/\alpha} \wedge \delta_D(x) \wedge \delta_D(y))^{\alpha/2} + m^{(2-\alpha)/\alpha}(\delta_D(x) \wedge \delta_D(y)) \\ \quad + m^{(2-\alpha)/(2\alpha)}(\delta_D(x) \wedge \delta_D(y))^{\alpha/2} \quad \text{when } |x-y| > 3m^{-1/\alpha}, \\ \big(\delta_D(x)\delta_D(y)\big)^{(\alpha-1)/2} \wedge \dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|}\mathbf{1}_{\{\delta_D(x)\wedge\delta_D(y) \le m^{-1/\alpha}\}} \\ \quad + m^{(2-\alpha)/\alpha}(\delta_D(x) \wedge \delta_D(y))\mathbf{1}_{\{\delta_D(x)\wedge\delta_D(y) > m^{-1/\alpha}\}} \\ \quad \text{when } |x-y| \le 3m^{-1/\alpha}; \end{cases}$$

when $d = 1 = \alpha$,

$$\widetilde{V}_D^{\alpha,m}(x,y) := \begin{cases} \dfrac{e^{-m|x-y|}}{|x-y|^{1/2}}(m^{-1} \wedge \delta_D(x) \wedge \delta_D(y))^{1/2} + m(\delta_D(x) \wedge \delta_D(y)) \\ \quad + m^{1/2}(\delta_D(x) \wedge \delta_D(y))^{1/2} \quad \text{when } |x-y| > 3m^{-1}, \\ \log\left(1 + \dfrac{\delta_D(x)^{1/2}\delta_D(y)^{1/2}}{|x-y|}\right) + m^{1/2}(\delta_D(x) \wedge \delta_D(y))\mathbf{1}_{\{\delta_D(x)\wedge\delta_D(y) > m^{-1}\}} \\ \quad \text{when } |x-y| \le 3m^{-1}; \end{cases}$$

and when $d = 1 > \alpha$,

$$\widetilde{V}_D^{\alpha,m}(x,y) := \begin{cases} \dfrac{m^{-1/2}e^{-m^{1/\alpha}|x-y|}}{|x-y|^{1-(\alpha/2)}}\left(1 \wedge \dfrac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2} + m^{(2-\alpha)/\alpha}(\delta_D(x) \wedge \delta_D(y)) \\ \quad + m^{(2-\alpha)/(2\alpha)}(\delta_D(x) \wedge \delta_D(y))^{\alpha/2} \quad \text{when } |x-y| > 3m^{-1/\alpha}, \\ |x-y|^{\alpha-1}\left(1 \wedge \dfrac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2} \\ \quad + m^{(2-\alpha)/\alpha}(\delta_D(x) \wedge \delta_D(y))\mathbf{1}_{\{\delta_D(x)\wedge\delta_D(y) > m^{-1/\alpha}\}} \\ \quad \text{when } |x-y| \le 3m^{-1/\alpha}. \end{cases}$$

Theorem 1.3. *Let $M > 0$ be a constant and D a half-space-like $C^{1,1}$ open set in $\mathbb{R}^d$ with $C^{1,1}$ characteristics (R, Λ_0). After an isometry, we may assume without loss of generality that $H_a \subset D \subset H_b$ for some real numbers $a > b$. Then there exist positive constants $C_7 < C_8$ depending only on d, α, R, Λ_0, $a - b$ and M such that for any $m \in (0, M]$ and $(x, y) \in D \times D$,*

$$C_7 \widetilde{V}_D^{\alpha,m}(x, y) \le G_D^m(x, y) \le C_8 \widetilde{V}_D^{\alpha,m}(x, y).$$

When D is the half space $H := \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_d > 0\}$ and $d \ge 2$, the two-sided estimates on G_H^1 were essentially obtained [17, Theorem 5.3]. However there is an error in the statement of [17, Theorem 5.3] for the case of $|x - y| \le 3$; see the proof of Theorem 3.1 below for details, which corrects the error. When $d = 1$ and $H := (0, \infty)$, a variant version of the two-sided estimates on G_H^1 was obtained in [17, Theorems 2.13 and 3.2]. We rewrite it to an equivalent form in the proof of Theorem 3.1 below and then derive from it the current version of the estimates on G_H^1 as stated in Theorem 1.3.

We note that the scaling relations (2.6)–(2.7) cannot reduce the proof of Theorem 1.3 to the case $m = 1$. This is because for a half-space-like $C^{1,1}$ open set D with $C^{1,1}$ characteristics (R, Λ_0), $m^{1/\alpha} D$ is, in general, a half-space-like $C^{1,1}$ open set with $C^{1,1}$ characteristics $(m^{1/\alpha} R, m^{-1/\alpha} \Lambda_0)$, which tends to $(0, \infty)$ as $m \to 0$.

The rest of the paper is organized as follows. In Section 2 we recall some basic facts about X^m and prove some preliminary uniform results on X^m, such as the uniform Harnack inequality and the uniform boundary Harnack principle. The proof of Theorem 1.3 is given in Section 3.

In the rest of this paper, we assume that $m > 0$. We will use capital letters $C_1, C_2, \ldots$ to denote constants in the statements of results, and their labeling will be fixed. The lower case constants $c_1, c_2, \ldots$ will denote generic constants used in proofs, whose exact values are not important and can change from one appearance to another. The labeling of the lower case constants starts anew in every proof. The dependence of the lower case constants on the dimension d will not always be mentioned explicitly. We will use ∂ to denote a cemetery point and for every function f, we extend its definition to ∂ by setting $f(\partial) = 0$. We will use dx to denote the Lebesgue measure in $\mathbb{R}^d$.

2. Uniform boundary Harnack principle

The Lévy measure of the relativistic α-stable process X^m, defined in (1.1), has a density

$$J^m(x) = j^m(|x|) := \frac{\alpha}{2\Gamma\left(1 - \frac{\alpha}{2}\right)} \int_0^\infty (4\pi u)^{-d/2} e^{-\frac{|x|^2}{4u}} e^{-m^{2/\alpha} u} u^{-(1+\frac{\alpha}{2})} du,$$

which is continuous and radially decreasing on $\mathbb{R}^d \setminus \{0\}$ (see [26, Lemma 2]). Here and in the rest of this paper Γ is the Gamma function defined by $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1} e^{-t} dt$ for every $\lambda > 0$. Put $J^m(x, y) := j^m(|x - y|)$. Let $\mathcal{A}(d, -\alpha) := \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma(\frac{d+\alpha}{2}) \Gamma(1 - \frac{\alpha}{2})^{-1}$. Using change of variables twice, first with $u = |x|^2 v$ then with $v = 1/s$, we get

$$J^m(x, y) = \mathcal{A}(d, -\alpha)|x - y|^{-d-\alpha} \psi(m^{1/\alpha}|x - y|) \tag{2.1}$$

where

$$\psi(r) := 2^{-(d+\alpha)} \Gamma\left(\frac{d+\alpha}{2}\right)^{-1} \int_0^\infty s^{\frac{d+\alpha}{2}-1} e^{-\frac{s}{4} - \frac{r^2}{s}} ds,$$

which is a decreasing smooth function of r^2 satisfying $\psi(0) = 1$, $\psi(r) \leq 1$ and

$$c_1^{-1} e^{-r} r^{(d+\alpha-1)/2} \leq \psi(r) \leq c_1 e^{-r} r^{(d+\alpha-1)/2} \quad \text{on } [1, \infty) \tag{2.2}$$

for some $c_1 > 1$ (see [11, pp. 276–277] for details). Recall that $X = X^0$ is a symmetric α-stable process and we denote the Lévy density of X by

$$J(x, y) := J^0(x, y) = \mathcal{A}(d, -\alpha)|x - y|^{-(d+\alpha)}.$$

The Lévy density gives rise to a Lévy system for X^m, which describes the jumps of the process X^m: for any nonnegative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s, y, y) = 0$ for all $y \in \mathbb{R}^d$, $x \in \mathbb{R}^d$ and stopping time T (with respect to the filtration of X^m),

$$\mathbb{E}_x \left[\sum_{s \leq T} f(s, X^m_{s-}, X^m_s) \right] = \mathbb{E}_x \left[\int_0^T \left(\int_{\mathbb{R}^d} f(s, X^m_s, y) J^m(X^m_s, y) dy \right) ds \right]. \tag{2.3}$$

(See, for example, [9, Proof of Lemma 4.7] and [10, Appendix A].)

The next two inequalities, which can be seen easily from (2.2), will be used later in this paper. For any $a > 0$ and $M > 0$, there exist positive constants C_9 and C_{10} depending on a and M such that for any $m \in (0, M]$,

$$j^m(r) \leq C_9 j^m(2r) \quad \text{for every } r \in (0, a) \tag{2.4}$$

and

$$j^m(r) \leq C_{10} j^m(r + a) \quad \text{for every } r > a. \tag{2.5}$$

We will use $p^m(t, x, y) = p^m(t, x - y)$ to denote the transition density of X^m and use $p(t, x, y)$ to denote the transition density of X.

For any open set D, we use τ^m_D to denote the first exit time from D for X^m, i.e., $\tau^m_D = \inf\{t > 0 : X^m_t \notin D\}$ and use τ_D to denote the first exit time from D for X. We define $X^{m,D}$ by $X^{m,D}_t(\omega) = X^m_t(\omega)$ if $t < \tau^m_D(\omega)$ and $X^{m,D}_t(\omega) = \partial$ if $t \geq \tau^m_D(\omega)$. We define X^D similarly. $X^{m,D}$ is called the subprocess of X^m in D (or, the killed relativistic α-stable process in D with mass m), and X^D is called the killed symmetric α-stable process in D.

It is known (see [10]) that $X^{m,D}$ has a transition density $p^m_D(t, x, y)$ which is continuous on $(0, \infty) \times D \times D$ with respect to the Lebesgue measure. Note that the transition density $p^m_D(t, x, y)$ may not be continuous on $\overline{D} \times \overline{D}$ if the boundary of D is irregular.

We will use $G^m_D(x, y) := \int_0^\infty p^m_D(t, x, y) dt$ to denote the Green function of $X^{m,D}$. We use $p_D(t, x, y)$ and $G_D(x, y)$ to denote the transition density and the Green function of X^D respectively.

From (1.1), one can easily see that X^m has the following scaling property:

$$\{m^{-1/\alpha}(X^1_{mt} - X^1_0), t \geq 0\} \quad \text{has the same distribution as that of } \{X^m_t - X^m_0, t \geq 0\},$$

i.e,

$$p^m(t, x, y) = m^{d/\alpha} p^1(mt, m^{1/\alpha} x, m^{1/\alpha} y).$$

Consequently,

$$p^m_D(t, x, y) = m^{d/\alpha} p^1_{m^{1/\alpha} D}(mt, m^{1/\alpha} x, m^{1/\alpha} y) \quad \text{for } t > 0 \text{ and } x, y \in D, \tag{2.6}$$

and

$$G_D^m(x, y) = m^{(d-\alpha)/\alpha} G^1_{m^{1/\alpha} D}(m^{1/\alpha} x, m^{1/\alpha} y) \quad \text{for every } x, y \in D. \tag{2.7}$$

The following result is established in [8, Theorem 2.6] (see also [17, (2.16)] and the comments following it).

Theorem 2.1. *There exist positive constants R_0 and $C_{11} > 1$ depending only on d and α such that for any $m \in (0, \infty)$, any ball B of radius $r \le R_0 m^{-1/\alpha}$,*

$$C_{11}^{-1} G_B(x, y) \le G_B^m(x, y) \le C_{11} G_B(x, y), \quad x, y \in B.$$

In the remainder of this paper, R_0 will always stand for the constant in Theorem 2.1.

A real-valued function u on $\mathbb{R}^d$ is said to be harmonic in an open set $D \subset \mathbb{R}^d$ with respect to X^m if for every open set B whose closure is a compact subset of D,

$$\mathbb{E}_x\left[\left|u(X^m_{\tau^m_B})\right|\right] < \infty \quad \text{and} \quad u(x) = \mathbb{E}_x\left[u(X^m_{\tau^m_B})\right] \quad \text{for every } x \in B.$$

A real-valued function u on $\mathbb{R}^d$ is said to be regular harmonic in an open set $D \subset \mathbb{R}^d$ with respect to X^m if

$$\mathbb{E}_x\left[\left|u(X^m_{\tau^m_D})\right|\right] < \infty \quad \text{and} \quad u(x) = \mathbb{E}_x\left[u(X^m_{\tau^m_D})\right] \quad \text{for every } x \in D.$$

Clearly, a regular harmonic function in D is harmonic in D. For any Greenian open set D and any $x \in D$, $G_D^m(\cdot, x)$ is harmonic in $D \setminus \{x\}$ with respect to X^m and regular harmonic in $D \setminus \overline{B(x, \varepsilon)}$ with respect to X^m for every $\varepsilon > 0$.

For any bounded open set U, define the Poisson kernel for X^m on U as

$$K_U^m(x, z) := \int_U G_U^m(x, y) J^m(y, z) dy, \quad (x, z) \in U \times (\mathbb{R}^d \setminus \overline{U}). \tag{2.8}$$

By Eq. (2.3),

$$\mathbb{E}_x\left[f(X^m_{\tau^m_U});\ X^m_{\tau^m_U -} \ne X^m_{\tau^m_U}\right] = \int_{\overline{U}^c} K_U^m(x, y) f(y) dy.$$

We will use K_U to denote the Poisson kernel of X on U. It is well known (see [1]) that for $x_0 \in \mathbb{R}^d$ and $r > 0$,

$$K_{B(x_0,r)}(x, z) = c_1 \frac{(r^2 - |x - x_0|^2)^{\frac{\alpha}{2}}}{(|z - x_0|^2 - r^2)^{\frac{\alpha}{2}}} \frac{1}{|x - z|^d} \quad \text{for all } (x, z) \in B(x_0, r) \times \overline{B(x_0, r)}^c \tag{2.9}$$

for some constant $c_1 = c_1(d, \alpha) > 0$.

The next proposition is essentially proved in [26, Theorem 6] for $m = 1$. We give a full proof using Theorem 2.1 for completeness.

Proposition 2.2. *There exist $C_{12}, C_{13} > 1$ depending only on d and α such that for all $m > 0$, $r \in (0, R_0 m^{-1/\alpha}]$ and $x_0 \in \mathbb{R}^d$,*

$$C_{12}^{-1} K_{B(x_0,r)}(x, y) \le K^m_{B(x_0,r)}(x, y) \le C_{12} K_{B(x_0,r)}(x, y) \tag{2.10}$$

on $B(x_0, r) \times \left(B(x_0, 2R_0 m^{-1/\alpha}) \setminus \overline{B(x_0, r)}\right)$, *and that for all* $U \subset B(x_0, r)$ *and* $(x, y) \in U \times (\mathbb{R}^d \setminus B(x_0, 3r/2))$,

$$C_{13}^{-1}\mathbb{E}_x[\tau_U^m] j^m(|y - x_0|) \le K_U^m(x, y) \le C_{13}\mathbb{E}_x[\tau_U^m] j^m(|y - x_0|). \tag{2.11}$$

In particular, there exists $C_{14} > 1$ *depending only on* d *and* α *such that on* $B(x_0, r) \times (\mathbb{R}^d \setminus B(x_0, 3r/2))$,

$$\begin{aligned} C_{14}^{-1} r^{\alpha/2}(r - |x - x_0|)^{\alpha/2} j^m(|y - x_0|) &\le K_{B(x_0,r)}^m(x, y) \\ &\le C_{14} r^{\alpha/2}(r - |x - x_0|)^{\alpha/2} j^m(|y - x_0|). \end{aligned} \tag{2.12}$$

Proof. Without loss of generality, we assume $x_0 = 0$. First of all, by (2.1), (2.8) and Theorem 2.1, $K_{B(x_0,r)}^m(x, y) \le c_1 K_{B(x_0,r)}(x, y)$ for $r \in (0, R_0 m^{-1/\alpha}]$. For the remainder of the proof, we assume $r \in (0, R_0 m^{-1/\alpha}]$.

For $z \in B(0, r)$ and $r < |y| < 2R_0 m^{-1/\alpha}$,

$$m^{1/\alpha}|z - y| \le m^{1/\alpha}(|z| + |y|) \le m^{1/\alpha}(r + |y|) \le 2m^{1/\alpha}|y| \le 4R_0, \tag{2.13}$$

thus, by (2.1), (2.8) and Theorem 2.1,

$$K_{B(x_0,r)}^m(x, y) \ge c_2\psi(4R_0)\int_{B(x_0,r)} G_{B(x_0,r)}(x, z)J(z, y)dz = c_2\psi(4R_0)K_{B(x_0,r)}(x, y).$$

On the other hand, for $z \in B(0, r)$ and $|y| \ge 2R_0 m^{-1/\alpha}$,

$$|m^{1/\alpha}y| - R_0 \le m^{1/\alpha}(|y| - |z|) \le m^{1/\alpha}|z - y| \le m^{1/\alpha}(|z| + |y|) \le R_0 + |m^{1/\alpha}y|.$$

Thus, by (2.5), for $z \in B(0, r)$ and $|y| \ge 2R_0 m^{-1/\alpha}$,

$$\begin{aligned} c_3\psi(|m^{1/\alpha}y|) &\le \psi(|m^{1/\alpha}y| + R_0) \le \psi(m^{1/\alpha}|z - y|) \\ &\le \psi(|m^{1/\alpha}y| - R_0) \le c_4\psi(|m^{1/\alpha}y|). \end{aligned}$$

Moreover, for $z \in B(0, r)$ and $3r/2 < |y| \le 2R_0 m^{-1/\alpha}$, using (2.13),

$$\frac{1}{3}m^{1/\alpha}|y| \le m^{1/\alpha}(|y| - |z|) \le m^{1/\alpha}(|z - y|) \le 2m^{1/\alpha}|y| \le 4R_0.$$

Thus, by (2.4), for $z \in B(0, r)$ and $3r/2 < |y| \le 2R_0 m^{-1/\alpha}$,

$$c_5\psi(|m^{1/\alpha}y|) \le \psi(2m^{1/\alpha}|y|) \le \psi(m^{1/\alpha}|z - y|) \le \psi\left(\frac{1}{3}m^{1/\alpha}|y|\right) \le c_6\psi(|m^{1/\alpha}y|).$$

Using these, (2.1), (2.8) and the fact that $(\frac{3}{5})^{d+\alpha}J(y) \le J(y, z) \le 3^{d+\alpha}J(y)$ for $z \in B(0, r)$ and $3r/2 < |y|$, we have

$$c_7\mathbb{E}_x[\tau_U^m] j^m(|y|) \le K_U^m(x, y) \le c_8\mathbb{E}_x[\tau_U^m] j^m(|y|).$$

We have proved (2.11).

Now (2.12) follows from (2.11), Theorem 2.1 and the estimates of $G_{B(0,r)}(x, y)$. □

The next proposition is proved in [8, Theorem 2.9] as a consequence of the uniform parabolic Harnack inequality established there. We give a different proof for completeness.

Theorem 2.3 (*Uniform Harnack Inequality*)**.** *There exists a constant $C_{15} = C_{15}(\alpha, d) > 0$ such that for any $m \in (0, \infty)$ and $r \in (0, 4m^{-1/\alpha}]$, $x_0 \in \mathbb{R}^d$ and any function u which is nonnegative in $\mathbb{R}^d$ and harmonic in $B(x_0, r)$ with respect to X^m we have*

$$u(x) \le C_{15} u(y) \quad \textit{for all } x, y \in B(x_0, r/2).$$

Proof. Since X^m satisfies the hypothesis **H** in [31], by [31, Theorem 1] we have $\mathbb{P}_x(X^m_{\tau^m_{B(x_0,r)}} \in \partial B(x_0, r)) = 0$, thus for $x \in B(x_0, r)$,

$$u(x) = \int_{B(x_0,r)^c} u(y) K^m_{B(x_0,r)}(x, y) dy.$$

Thus for $r \in (0, R_0 m^{-1/\alpha}]$, the theorem follows from (2.9) and Proposition 2.2. When $r \in (R_0 m^{-1/\alpha}, 4m^{-1/\alpha}]$, we apply the usual chain argument to get the conclusion of the theorem. □

To get our uniform Green function estimates in half-space-like open sets, we need a uniform boundary Harnack principle (Theorem 2.6 below). We emphasize that Theorem 2.6 is not a direct consequence of the boundary Harnack principle in [22].

Lemma 2.4. *For every $\kappa \in (0, 1)$, there exists $C_{16} = C_{16}(\kappa, d, \alpha) > 0$ such that for any $m \in (0, \infty)$, $r \in (0, \frac{1}{2} R_0 m^{-1/\alpha}]$, any open set U with $B(A, \kappa r) \subset U \subset B(z, r)$ and $x \in U \cap B(z, \frac{r}{2})$, we have*

$$\mathbb{P}_x\left(X^m_{\tau^m_U} \in B(z, r)^c\right) \le C_{16} \mathbb{P}_x\left(X^m_{\tau^m_{U \setminus B(A,\kappa r)}} \in B(A, \kappa r)\right).$$

Proof. The proof of this lemma is similar to that of [29, Lemma 3.3]. We spell out the details here for the readers' convenience. Without loss of generality, we assume that $z = 0$. We fix $x \in U \setminus B(A, \kappa r)$ until the display (2.16) and let $B := B(A, \frac{\kappa r}{2})$. Since $G^m_U(x, \cdot)$ is harmonic in $U \setminus \{x\}$ with respect to X^m,

$$G^m_U(x, A) = \int_{U \cap B^c} K^m_B(A, y) G^m_U(x, y) dy \ge \int_{U \cap B(A, \frac{3\kappa r}{4})^c} K^m_B(A, y) G^m_U(x, y) dy.$$

Since $\frac{3\kappa r}{4} \le |y - A| \le 2r \le R_0 m^{-1/\alpha}$ for $y \in B(A, \frac{3\kappa r}{4})^c \cap U$, it follows from (2.9) and (2.10) that

$$\begin{aligned} G^m_U(x, A) &\ge c_1 \int_{U \cap B(A, \frac{3\kappa r}{4})^c} \frac{(\kappa r)^\alpha}{|y - A|^{d+\alpha}} G^m_U(x, y) dy \\ &\ge c_2 r^{-d} \int_{U \cap B(A, \frac{3\kappa r}{4})^c} G^m_U(x, y) dy. \end{aligned}$$

Using this and applying Theorem 2.3 we get

$$\int_U G^m_U(x, y) dy \le c_3 r^d G^m_U(x, A) + \int_{B(A, \frac{3\kappa r}{4})} G^m_U(x, y) dy \le c_4 r^d G^m_U(x, A).$$

Let $V = U \setminus B(A, \frac{\kappa r}{2})$. Note that for any $w \in B(A, \frac{\kappa r}{4})$ and $y \in V$,

$$2^{-1}|y - w| \le |y - A| \le 2|y - w| \quad \text{and} \quad m^{1/\alpha}|y - A| \le 2m^{1/\alpha}|y - w| \le 2R_0.$$

Thus we get from (2.8) that for $w \in B(A, \frac{\kappa r}{4})$,

$$c_5 K^m_V(x, A) \le K^m_V(x, w) \le c_6 K^m_V(x, A). \tag{2.14}$$

Using the harmonicity of $G_U^m(\cdot, A)$ in $U \setminus \{A\}$ with respect to X^m, we can split $G_U^m(\cdot, A)$ into two parts:

$$\begin{aligned} G_U^m(x, A) &= \mathbb{E}_x\left[G_U^m(X^m_{\tau^m_V}, A)\right] \\ &= \mathbb{E}_x\left[G_U^m(X^m_{\tau^m_V}, A) : X^m_{\tau^m_V} \in B(A, \kappa r/4)\right] \\ &\quad + \mathbb{E}_x\left[G_U^m(X^m_{\tau^m_V}, A) : X^m_{\tau^m_V} \in \{\kappa r/4 \le |y - A| \le \kappa r/2\}\right] \\ &:= I_1 + I_2. \end{aligned}$$

Since $2r \le m^{-1/\alpha} R_0$, by Theorem 2.1, and the monotonicity and symmetry of the Green functions, we have

$$G_U^m(y, A) \le G^m_{B(0,r)}(y, A) \le G^m_{B(A,2r)}(y, A) \le c_7 G_{B(A,2r)}(A, y) \quad y \in B(A, \kappa r/2). \tag{2.15}$$

Thus

$$\int_{B(A, \frac{\kappa r}{4})} G_U^m(y, A) dy \le c_7 \int_{B(A, \frac{\kappa r}{4})} G_{B(A,2r)}(A, y) \le c_8 r^\alpha.$$

Using this and (2.14) twice, we have

$$\begin{aligned} I_1 &\le c_9 K_V^m(x, A) \int_{B(A, \frac{\kappa r}{4})} G_U^m(y, A) dy \le c_{10} r^\alpha K_V^m(x, A) \\ &\le c_{11} r^{\alpha - d} \int_{B(A, \frac{\kappa r}{4})} K_V^m(x, z) dz. \end{aligned}$$

On the other hand, since by (2.15) and [7, Corollary 1.2] $G_U^m(y, A) \le c_{12} r^{\alpha - d}$ on $y \in \{\frac{\kappa r}{4} \le |y - A| \le \frac{\kappa r}{2}\}$ for all $d \ge 1$, we get

$$\begin{aligned} I_2 &\le \int_{\{\frac{\kappa r}{4} \le |y-A| \le \frac{\kappa r}{2}\}} G^m_{B(A,2r)}(A, y) \mathbb{P}_x(X^m_{\tau^m_V} \in dy) \\ &\le c_{12} r^{\alpha - d} \mathbb{P}_x\left(X^m_{\tau^m_V} \in \left\{\frac{\kappa r}{4} \le |y - A| \le \frac{\kappa r}{2}\right\}\right). \end{aligned}$$

Therefore

$$\begin{aligned} \int_U G_U^m(x, y) dy &\le c_{13} r^\alpha \mathbb{P}_x\left(X^m_{\tau^m_{U \setminus B\left(A, \frac{\kappa r}{2}\right)}} \in B\left(A, \frac{\kappa r}{2}\right)\right) \\ &\le c_{13} r^\alpha \mathbb{P}_x\left(X^m_{\tau^m_{U \setminus B(A, \kappa r)}} \in B(A, \kappa r)\right). \end{aligned} \tag{2.16}$$

Recall that $C_c^\infty(\mathbb{R}^d)$, the space of continuous functions with compact support, is in the domain of the L_2-generator $\mathcal{L}_m$ of X^m and

$$\mathcal{L}_m \phi(x) = \int_{\mathbb{R}^d} (\phi(x + y) - \phi(x) - (\nabla \phi(x) \cdot y) 1_{B(0,\varepsilon)}(y)) J^m(|y|) dy, \quad \forall \phi \in C_c^\infty(\mathbb{R}^d)$$

(see [27, Section 4.1]). Using the argument in [21, pp. 152], one can easily see that the last formula on [21, pp. 152] is valid for X^m for all $d \ge 1$. Thus we have that for every $x \in U$ and ϕ

in $C_c^\infty(\mathbb{R}^d)$ with $\phi(x) = 0$,

$$\mathbb{E}_x\left[\phi(X^m_{\tau^m_U})\right] = \int_U G^m_U(x, y)\mathcal{L}_m\phi(y)dy. \tag{2.17}$$

Take a sequence of radial functions ϕ_k in $C_c^\infty(\mathbb{R}^d)$ such that $0 \le \phi_k \le 1$,

$$\phi_k(y) = \begin{cases} 0, & |y| < 1/2 \\ 1, & 1 \le |y| \le k+1 \\ 0, & |y| > k+2, \end{cases}$$

and that $\sum_{i,j} |\frac{\partial^2}{\partial y_i \partial y_j}\phi_k|$ is uniformly bounded. Define $\phi_{k,r}(y) = \phi_k(\frac{y}{r})$. Then we have $0 \le \phi_{k,r} \le 1$ and

$$\sup_{y\in\mathbb{R}^d} \sum_{i,j} \left|\frac{\partial^2}{\partial y_i \partial y_j}\phi_{k,r}(y)\right| < c_{14} r^{-2}.$$

Using this inequality and the fact that $J^m \le J$, we have

$$\begin{aligned}
\sup_{k\ge1}\sup_{x\in\mathbb{R}^d} |\mathcal{L}_m\phi_{k,r}(x)| &\le \sup_{k\ge1}\sup_{x\in\mathbb{R}^d} \bigg|\int_{\mathbb{R}^d}(\phi_{k,r}(x+y) - \phi_{k,r}(x) \\
&\quad - (\nabla\phi_{k,r}(x)\cdot y)1_{B(0,r)}(y))j^m(|y|)dy\bigg| \\
&\le c_{15}\sup_{k\ge1}\sup_{x\in\mathbb{R}^d}\left(\int_{\{|y|\le r\}}\left|\frac{\phi_{k,r}(x+y) - \phi_{k,r}(x) - (\nabla\phi_{k,r}(x)\cdot y)}{|y|^{d+\alpha}}\right|dy \right.\\
&\quad \left. + \int_{\{r<|y|\}}\frac{1}{|y|^{d+\alpha}}dy\right) \\
&\le c_{16}\left(\frac{1}{r^2}\int_{\{|y|\le r\}}\frac{|y|^2}{|y|^{d+\alpha}}dy + \int_{\{r<|y|\}}\frac{1}{|y|^{d+\alpha}}dy\right) \le c_{17}r^{-\alpha}. \qquad (2.18)
\end{aligned}$$

When $U \subset B(0, r)$ for some $r \in (0, 1)$, we get, by combining (2.17) and (2.18), that for any $x \in U \cap B(0, \frac{r}{2})$,

$$\begin{aligned}
\mathbb{P}_x\left(X^m_{\tau^m_U} \in B(0, r)^c\right) &= \lim_{k\to\infty}\mathbb{P}_x\left(X^m_{\tau^m_U} \in \{y \in \mathbb{R}^d : r \le |y| < (k+1)r\}\right) \\
&\le c_{18} r^{-\alpha}\int_U G^m_U(x, y)dy.
\end{aligned}$$

Combining this with (2.16), we have proved the lemma. □

Lemma 2.5. *Suppose $m \in (0, \infty)$, $\kappa \in (0, \frac{1}{2}]$, $r \in (0, \frac{1}{2}R_0 m^{-1/\alpha}]$ and that D is an open set with $B(A, \kappa r) \subset D \cap B(z, r)$. Suppose that $u \ge 0$ is regular harmonic in $D \cap B(z, 2r)$ with respect to X^m and $u = 0$ in $D^c \cap B(z, 2r)$. If w is a regular harmonic function with respect to X^m in $D \cap B(z, r)$ such that*

$$w(x) = \begin{cases} u(x), & x \in B(z, 3r/2)^c \cup (D^c \cap B(z, r)), \\ 0, & x \in \{y \in \mathbb{R}^d : r \le |y - z| < 3r/2\}, \end{cases}$$

then

$$u(A) \geq w(A) \geq C_{17} u(x) \quad \textit{for } x \in D \cap B(z, 3r/2)$$

for some constant $C_{17} > 0$ *depending only on* d, α, κ.

Proof. Without loss of generality, we may assume $z = 0$ and $x \in D \cap B(0, \frac{3}{2}r)$. The left-hand side inequality in the conclusion of the lemma is obvious, so we only need to prove the right-hand side inequality.

Using (2.11) and then applying Theorem 2.1 and the Green function estimates of X in balls, we have that

$$\begin{aligned} w(A) &= \int_{B(0,\frac{3r}{2})^c} K^m_{D\cap B(0,r)}(A, y) u(y) dy \geq c_1 \int_{B(0,\frac{3r}{2})^c} \mathbb{E}_A[\tau^m_{D\cap B(0,r)}] j^m(|y|) u(y) dy \\ &\geq c_2 \int_{B(0,\frac{3r}{2})^c} j^m(|y|) \mathbb{E}_A[\tau^m_{B(A,\kappa r)}] u(y) dy \geq c_3 r^\alpha \int_{B(0,\frac{3r}{2})^c} J^m(y) u(y) dy. \end{aligned} \tag{2.19}$$

Note that

$$\begin{aligned} &\int_{\frac{10}{6}r}^{\frac{11}{6}r} \int_{\{y\in\mathbb{R}^d : s\leq |y|<2r\}} (|y|-s)^{-\alpha/2} u(y) dy ds \\ &= \int_{\{y\in\mathbb{R}^d : \frac{10}{6}r\leq |y|<2r\}} \int_{\frac{10}{6}r}^{|y|\wedge\frac{11}{6}r} (|y|-s)^{-\alpha/2} ds u(y) dy \\ &\leq \int_{\{y\in\mathbb{R}^d : \frac{10}{6}r\leq |y|<2r\}} \left(\int_0^{|y|-\frac{10}{6}r} s^{-\alpha/2} ds \right) u(y) dy \\ &\leq \frac{2}{(2-\alpha)3^{1-\alpha/2}} r^{1-\alpha/2} \int_{\{y\in\mathbb{R}^d : \frac{10}{6}r\leq |y|<2r\}} u(y) dy. \end{aligned}$$

Thus, there is an $s \in (\frac{10}{6}r, \frac{11}{6}r)$ such that

$$\begin{aligned} &\int_{\{y\in\mathbb{R}^d : s\leq |y|<2r\}} (|y|-s)^{-\alpha/2} u(y) dy \\ &\leq \frac{12}{(2-\alpha)3^{1-\alpha/2}} r^{-\alpha/2} \int_{\{y\in\mathbb{R}^d : \frac{10}{6}r\leq |y|<2r\}} u(y) dy. \end{aligned} \tag{2.20}$$

Let $x \in D \cap B(0, \frac{3}{2}r)$. Note that, since X^m satisfies the hypothesis **H** in [31], by [31, Theorem 1], we have

$$\begin{aligned} u(x) &= \mathbb{E}_x \left[u(X^m_{\tau^m_{D\cap B(0,s)}}); X^m_{\tau^m_{D\cap B(0,s)}} \in B(0,s)^c \right] \\ &= \mathbb{E}_x \left[u(X^m_{\tau^m_{B(0,s)}}); X^m_{\tau^m_{B(0,s)}} \in B(0,s)^c, \tau^m_{D\cap B(0,s)} = \tau^m_{B(0,s)} \right] \\ &\leq \mathbb{E}_x \left[u(X^m_{\tau^m_{B(0,s)}}); X^m_{\tau^m_{B(0,s)}} \in B(0,s)^c \right] = \int_{B(0,s)^c} K^m_{B(0,s)}(x, y) u(y) dy. \end{aligned}$$

In the first equality above we have used the fact that u vanishes on $D^c \cap B(0, s)$. Since

$$|x| < \frac{3}{2}r < \frac{10}{6}r < s < \frac{11}{6}r < 2r < R_0 m^{-1/\alpha},$$

from Proposition 2.2 and (2.9) we have

$$\begin{aligned} u(x) &\leq c_4 \int_{\{y\in\mathbb{R}^d:s\leq|y|<2r\}} K_{B(0,s)}(x,y)u(y)dy \\ &\quad + c_4 \int_{B(0,2r)^c} j^m(|y|)s^{\alpha/2}(s-|x|)^{\alpha/2}u(y)dy \\ &\leq c_5 r^{\alpha/2-d} \int_{\{y\in\mathbb{R}^d:s\leq|y|<2r\}} (|y|-s)^{-\alpha/2}u(y)dy + c_5 r^{\alpha} \int_{B(0,2r)^c} j^m(|y|)u(y)dy \end{aligned}$$

for some constant $c_4, c_5 > 0$. By (2.20),

$$\begin{aligned} & r^{\alpha/2-d} \int_{\{y\in\mathbb{R}^d:s\leq|y|<2r\}} (|y|-s)^{-\alpha/2}u(y)dy \\ &\quad \leq \frac{12}{(2-\alpha)3^{1-\alpha/2}} r^{-d} \int_{\{y\in\mathbb{R}^d:\frac{10}{6}r\leq|y|<2r\}} u(y)dy \\ &\quad \leq \frac{12\cdot 2^{d+\alpha}}{(2-\alpha)3^{1-\alpha/2}} r^{\alpha} \int_{\{y\in\mathbb{R}^d:\frac{10}{6}r\leq|y|<2r\}} |y|^{-d-\alpha}u(y)dy. \end{aligned}$$

Hence, combining this with Eq. (2.19)

$$u(x) \leq c_6 r^{\alpha} \int_{B(0,\frac{10r}{6})^c} J^m(y)u(y)dy \leq c_7 w(A). \quad \square$$

Using Lemmas 2.4 and 2.5, we can repeat the argument in the proof of the boundary Harnack principle in [2,22,29] to arrive at our uniform boundary Harnack principle. We spell out the details for the readers' convenience.

Theorem 2.6 (*Uniform Boundary Harnack Principle*)**.** *Suppose that D is an open set in $\mathbb{R}^d$ and $\kappa \in (0, \frac{1}{2}]$. There exists $C_{18} = C_{18}(d,\alpha,\kappa) > 0$ such that for all $m \in (0,\infty)$, $z \in \partial D$, $r \in (0, \frac{1}{2}R_0 m^{-1/\alpha}]$, $B(A,\kappa r) \subset D\cap B(z,r)$ and all functions $u, v \geq 0$ on $\mathbb{R}^d$ which are positive regular harmonic for X^m in $D\cap B(z,2r)$ and vanishing on $D^c\cap B(z,2r)$, we have*

$$C_{18}^{-1}\frac{u(A)}{v(A)} \leq \frac{u(x)}{v(x)} \leq C_{18}\frac{u(A)}{v(A)}, \quad x \in D\cap B(z,r).$$

Proof. Fix $r \in (0, \frac{1}{2}R_0 m^{-1/\alpha}]$ throughout this proof. Without loss of generality, we may assume that $z = 0$ and $u(A) = v(A)$. Define

$$\begin{aligned} u(x) &= \mathbb{E}_x[u(X^m_{\tau^m_{D\cap B(0,r)}}) : X^m_{\tau^m_{D\cap B(0,r)}} \in \{y\in\mathbb{R}^d : r\leq|y|<3r/2\}] \\ &\quad + \int_{B(0,\frac{3r}{2})^c} K^m_{D\cap B(0,r)}(x,z)u(z)dz \\ &=: u_1(x) + u_2(x). \end{aligned} \tag{2.21}$$

If $D\cap\{y\in\mathbb{R}^d : r\leq|y|<3r/2\}$ is empty, then, since u vanishes on $D^c\cap B(0,2r)$, $u_1 = 0$ and the inequality (2.24) below holds trivially. So we assume $D\cap\{y\in\mathbb{R}^d : r\leq|y|<3r/2\}$ is not empty. Then by Lemmas 2.4 and 2.5, for $x \in D \cap B(0, \frac{r}{2})$, we have

$$
\begin{aligned}
u_1(x) &\le \left(\sup_{D\cap\{y\in\mathbb{R}^d : r\le |y|<3r/2\}} u(y) \right) \mathbb{P}_x \left(X^m_{\tau^m_{D\cap B(0,r)}} \in B(0,r)^c \right) \\
&\le c_1 u(A) \mathbb{P}_x \left(X^m_{\tau^m_{D\cap B(0,r)}} \in B(0,r)^c \right) \\
&\le c_2 u(A) \mathbb{P}_x \left(X^m_{\tau^m_{(D\cap B(0,r))\setminus B\left(A,\frac{\kappa r}{2}\right)}} \in B\left(A, \frac{\kappa r}{2}\right) \right). \qquad (2.22)
\end{aligned}
$$

Since $2r < R_0 m^{-1/\alpha}$, by Theorem 2.3, for $x \in D \cap B(0, \frac{r}{2})$,

$$
\begin{aligned}
u(x) &= \mathbb{E}_x \left[u(X^m_{\tau^m_{(D\cap B(0,r))\setminus B\left(A,\frac{\kappa r}{2}\right)}}) \right] \\
&\ge c_3 u(A) \mathbb{P}_x \left(X^m_{\tau^m_{(D\cap B(0,r))\setminus B\left(A,\frac{\kappa r}{2}\right)}} \in B\left(A, \frac{\kappa r}{2}\right) \right). \qquad (2.23)
\end{aligned}
$$

Using (2.22), the analogue of (2.23) for v and the assumption that $u(A) = v(A)$, we get that for $x \in D \cap B(0, \frac{r}{2})$,

$$
u_1(x) \le c_2 v(A) \mathbb{P}_x \left(X^m_{\tau^m_{(D\cap B(0,r))\setminus B\left(A,\frac{\kappa r}{2}\right)}} \in B\left(A, \frac{\kappa r}{2}\right) \right) \le c_4 v(x) \qquad (2.24)
$$

for some constant $c_4 = c_4(\kappa) > 0$.

On the other hand, by (2.11), for $x \in D \cap B(0, r)$, we have

$$
\begin{aligned}
c_5 \mathbb{E}_x \left[\tau^m_{D\cap B(0,r)} \right] \int_{B(0,\frac{3r}{2})^c} j^m(|z|) u(z) dz &\le u_2(x) \\
\le c_6 \mathbb{E}_x \left[\tau^m_{D\cap B(0,r)} \right] \int_{B(0,\frac{3r}{2})^c} j^m(|z|) u(z) dz.
\end{aligned}
$$

Thus, let $s(x) = \mathbb{E}_x[\tau^m_{D\cap B(0,r)}]$, we have

$$
c_7^{-1} \le \frac{u_2(x)}{u_2(A)} \Big/ \frac{\mathbb{E}_x[\tau^m_{D\cap B(0,r)}]}{\mathbb{E}_A[\tau^m_{D\cap B(0,r)}]} \le c_7, \qquad (2.25)
$$

for some constant $c_7 > 1$. Applying (2.25) to u and v and Lemma 2.5 to v and v_2, we obtain for $x \in D \cap B(0, \frac{r}{2})$,

$$
u_2(x) \le c_7 u_2(A) \frac{s(x)}{s(A)} \le c_7^2 \frac{u_2(A)}{v_2(A)} v_2(x) \le c_8 \frac{u(A)}{v(A)} v_2(x) = c_8 v_2(x) \qquad (2.26)
$$

for some constant $c_8 > 0$. Combining (2.21), (2.24) and (2.26), we have

$$
u(x) \le c_9 v(x) \quad \text{for every } x \in D \cap B(0, r/2). \quad \square
$$

3. Green function estimates

In this section, we present the proof of Theorem 1.3.

Theorem 3.1. *Let $H := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_d > 0\}$ be the upper half space in $\mathbb{R}^d$. There exists $C_{19} = C_{19}(d, \alpha) > 1$ such that for all $m > 0$,*

$$C_{19}^{-1} \widetilde{V}_H^{\alpha,m}(x, y) \le G_H^m(x, y) \le C_{19} \widetilde{V}_H^{\alpha,m}(x, y), \quad x, y \in H,$$

where $\widetilde{V}_H^{\alpha,m}$ is defined before the statement of Theorem 1.3.

Proof. Since by (2.7),

$$G_H^m(x, y) = m^{(d-\alpha)/\alpha} G_H^1(m^{1/\alpha} x, m^{1/\alpha} y) \quad \text{for } x, y \in H, \tag{3.1}$$

it suffices to consider $m = 1$. When $m = 1$, the $d \ge 2$ case of this theorem is essentially established in [17, Theorem 5.3]. However there is an error in the statement of [17, Theorem 5.3] for the case of $|x - y| \le 3$, where the terms $\left(\frac{x_d \wedge y_d}{|x-y|}\right)^{\alpha/2}$ should be replaced by $\left(\frac{x_d y_d}{|x-y|^2}\right)^{\alpha/2}$. The error in [17, Theorem 5.3] stems from [5, Theorem 3.2], where the same error occurred in the estimate of the 1-resolvent of X^1 in the upper half space. [5, Theorem 3.2] is a corollary of [5, Lemma 3.1]. A typo occurred in the display preceding the statement of [5, Theorem 3.2], which resulted in all these errors. That display should be

$$\left(\sqrt{\frac{4\delta(x)\delta(y)}{|x-y|^2} + 1} - 1\right) |x - y| \approx \frac{\delta(x)\delta(y)}{|x-y|} \quad \text{for } \frac{\delta(x)\delta(y)}{|x-y|^2} \le 1.$$

Another typo occurred in [5, (21) and (22)], where the term $\delta(x) \wedge \delta(y) \wedge 1$ should be $\frac{\delta(x)\delta(y)}{|x-y|} \wedge 1$. With these corrections, the desired Green function estimates can then be established as in [17, Theorem 5.3].

Now we deal with the case $d = 1$. In the remainder of this proof the notation $f \asymp g$ means that there are positive constants c_1 and c_2 depending only on α so that $c_1 g(x) \le f(x) \le c_2 g(x)$ in the common domain of definition for f and g. We first recall the result in [17, Theorems 2.13 and 3.2].

(i) For $\alpha > 1$ and $x, y > 0$,

$$G_H^1(x, y) \asymp \begin{cases} \dfrac{e^{-|x-y|}}{|x-y|^{1-(\alpha/2)}} (1 \wedge x \wedge y)^{\alpha/2} + (x \wedge y) + (x \wedge y)^{\alpha/2} \\ \quad \text{when } |x - y| \ge 1 \wedge x \wedge y, \\ (1 \wedge x \wedge y)^{\alpha-1} + (x \wedge y) + (x \wedge y)^{\alpha/2} \\ \quad \text{when } |x - y| < 1 \wedge x \wedge y. \end{cases} \tag{3.2}$$

(ii) For $\alpha = 1$ and $x, y > 0$,

$$G_H^1(x, y) \asymp \begin{cases} \dfrac{e^{-|x-y|}}{|x-y|^{1/2}} (1 \wedge x \wedge y)^{1/2} + (x \wedge y) + (x \wedge y)^{1/2} \\ \quad \text{when } |x - y| \ge 1 \wedge x \wedge y, \\ \log\left(2 \dfrac{1 \wedge x \wedge y}{|x-y|}\right) + (x \wedge y) + (x \wedge y)^{1/2} \\ \quad \text{when } |x - y| < 1 \wedge x \wedge y. \end{cases} \tag{3.3}$$

(iii) For $0 < \alpha < 1$ and $x, y > 0$,

$$G_H^1(x,y) \asymp \begin{cases} \dfrac{e^{-|x-y|}}{|x-y|^{1-(\alpha/2)}}\left(1 \wedge \dfrac{xy}{|x-y|^2}\right)^{\alpha/2} + (x \wedge y) + (x \wedge y)^{\alpha/2} \\ \quad \text{when } |x-y| \ge 1, \\ |x-y|^{\alpha-1}\left(1 \wedge \dfrac{xy}{|x-y|^2}\right)^{\alpha/2} + (x \wedge y) + (x \wedge y)^{\alpha/2} \\ \quad \text{when } |x-y| < 1. \end{cases} \tag{3.4}$$

In the above estimates, we have slightly rewritten the result in [17, Theorems 2.13 and 3.2] to an equivalent form, which can be seen from the inequality

$$x < \frac{xy}{y-x} = x\left(1 + \frac{x}{y-x}\right) \le 2x \quad \text{for every } 0 < x < y \text{ with } xy \le (y-x)^2.$$

By (3.2)–(3.4), we only need to consider the case $|x-y| < 3$. So in the remainder of the proof, we assume that $x \le y$ and $|x-y| = y-x < 3$. We first deal with the case $\alpha = 1$. We consider three subcases separately.

(a) $x \le 1$ and $x \le y-x$: In this case, we have $y = x + (y-x) \le 2(y-x)$. Since $y-x < 3$, we have $\frac{x^{1/2}}{(y-x)^{1/2}} \ge \frac{1}{\sqrt{3}} x^{1/2}$. Thus by (3.3),

$$\frac{1}{\sqrt{2}}\frac{x^{1/2}y^{1/2}}{y-x} \le \frac{x^{1/2}}{(y-x)^{1/2}} \asymp G_H^1(x,y) \asymp \frac{x^{1/2}}{(y-x)^{1/2}} \le \frac{x^{1/2}y^{1/2}}{y-x}.$$

Since $\frac{x^{1/2}y^{1/2}}{y-x} \le 2$, we get

$$G_H^1(x,y) \asymp \log\left(1 + \frac{x^{1/2}y^{1/2}}{y-x}\right). \tag{3.5}$$

(b) $x \le 1$ and $y-x < x$: In this case, we have $x \le y \le 2x$ and

$$x \log 2 \le x^{1/2} \log 2 \le \log 2 \le \log\left(2\frac{x}{y-x}\right).$$

Thus by (3.3),

$$\begin{aligned} \log\left(\sqrt{2}\frac{x^{1/2}y^{1/2}}{y-x}\right) &\le \log\left(2\frac{x}{y-x}\right) \asymp G_H^1(x,y) \asymp \log\left(2\frac{x}{y-x}\right) \\ &\le \log\left(2\frac{x^{1/2}y^{1/2}}{y-x}\right). \end{aligned}$$

Since $\frac{x^{1/2}y^{1/2}}{y-x} \ge 1$, we get (3.5).

(c) $x > 1$: In this case, since $y-x < 3$ we have $x \le y \le 4x$. Thus by (3.3), if $y-x < 1$

$$\begin{aligned} &\log\left(2\frac{x^{1/2}y^{1/2}}{y-x}\right) + cx^{1/2}y^{1/2} \le \log\left(\frac{2}{y-x}\right) + \log(x^{1/2}y^{1/2}) + cx^{1/2}y^{1/2} \\ &\le \log\left(\frac{2}{y-x}\right) + x^{1/2}y^{1/2} \asymp G_H^1(x,y) \asymp \log\left(\frac{2}{y-x}\right) + x \\ &\le \log\left(2\frac{x^{1/2}y^{1/2}}{y-x}\right) + x. \end{aligned}$$

If $1 \le y - x < 3$, we have $\log\left(2\frac{x^{1/2}y^{1/2}}{y-x}\right) \le \log(4x) \le 4x$. Thus by (3.3),

$$x \le \frac{1}{(y-x)^{1/2}} + x \asymp G_H^1(x, y) \asymp \frac{1}{(y-x)^{1/2}} + x \le 1 + x \le 2x.$$

Since $\frac{x^{1/2}y^{1/2}}{y-x} \ge 1$ and $x > 1$, we get

$$G_H^1(x, y) \asymp \log\left(1 + \frac{x^{1/2}y^{1/2}}{y-x}\right) + (x \wedge y). \tag{3.6}$$

Now we consider the case $\alpha > 1$. Again we divide into three subcases.
(a) $x \le 1$ and $x \le y - x$: In this case, we have $(y - x) \le y \le 2(y - x) \wedge 4$ and so

$$\frac{x^{\alpha/2}}{(y-x)^{1-\alpha/2}} \ge \frac{1}{4^{1-\alpha/2}} x^{\alpha/2}.$$

Thus by (3.2),

$$\frac{x^{\alpha/2}}{y^{1-\alpha/2}} \le \frac{x^{\alpha/2}}{(y-x)^{1-\alpha/2}} \asymp G_H^1(x, y) \asymp \frac{x^{\alpha/2}}{(y-x)^{1-\alpha/2}} \le 2^{1-\alpha/2} \frac{x^{\alpha/2}}{y^{1-\alpha/2}}.$$

From this we immediately get

$$G_H^1(x, y) \asymp \left(x^{(\alpha-1)/2} y^{(\alpha-1)/2}\right) \wedge \frac{x^{\alpha/2}y^{\alpha/2}}{y-x}. \tag{3.7}$$

(b) $x \le 1$ and $y - x < x$: In this case, we have $x \le y \le 2x$, and by (3.2),

$$\frac{1}{2}\frac{x^{\alpha/2}}{y^{1-\alpha/2}} \asymp x^{\alpha-1} \asymp G_H^1(x, y) \asymp x^{\alpha-1} \le 2^{1-\alpha/2} \frac{x^{\alpha/2}}{y^{1-\alpha/2}}.$$

Again (3.7) follows immediately.
(c) $x > 1$: In this case, since $y - x < 3$ we have $x \le y \le 4x$, and by (3.2),

$$G_H^1(x, y) \asymp 1 + x \asymp x.$$

Now we consider the case $\alpha < 1$. The subcase $x > 1$ is clear from (3.4). Note that

$$(y-x)^{\alpha-1}\left(1 \wedge \frac{x^{\alpha/2}y^{\alpha/2}}{(y-x)^{\alpha}}\right) \asymp (y-x)^{\alpha-1} \frac{x^{\alpha/2}y^{\alpha/2}}{(x \vee y \vee (y-x))^{\alpha}} = (y-x)^{\alpha-1} \frac{x^{\alpha/2}}{y^{\alpha/2}}.$$

(a) $x \le 1$ and $x \le y - x$: In this case, we have $(y - x) \le y \le 2(y - x) \wedge 4$ and so

$$(y-x)^{\alpha-1} \frac{x^{\alpha/2}}{y^{\alpha/2}} \asymp \frac{x^{\alpha/2}}{(y-x)^{1-\alpha/2}} \ge \frac{1}{4^{1-\alpha/2}} x^{\alpha/2}.$$

Thus by (3.4),

$$G_H^1(x, y) \asymp (y-x)^{\alpha-1} \frac{x^{\alpha/2}}{y^{\alpha/2}} \asymp (y-x)^{\alpha-1}\left(1 \wedge \frac{x^{\alpha/2}y^{\alpha/2}}{(y-x)^{\alpha}}\right). \tag{3.8}$$

(b) $x \le 1$ and $y - x < x$: In this case, we have $x \le y \le 2x$, so

$$(y-x)^{\alpha-1} \frac{x^{\alpha/2}}{y^{\alpha/2}} \asymp (y-x)^{\alpha-1} \ge 1 \ge x^{\alpha/2}.$$

Thus (3.8) follows from (3.4). $\square$

As a consequence of this theorem, we can easily see that, for any $b > 1$, there exists a positive constant c such that

$$G^1_H(bx, by) \le cG^1_H(x, y) \quad \text{for all } x, y \in H. \tag{3.9}$$

The inequalities in the next lemma can be proved by elementary calculus and will be used several times without being mentioned explicitly.

Lemma 3.2. *For any $L > 0$, there exists a constant $C_{20} = C_{20}(L) > 1$ such that*

$$C_{20}^{-1} b \le \log(1+b) \le b \quad \textit{for any } 0 < b \le L$$

and

$$C_{20}^{-1} \log(1+s) \le \log(1+Ls) \le C_{20} \log(1+s) \quad \textit{for any } 0 < s < \infty.$$

We will also use the following fact several times in this section.

$$\begin{aligned} \frac{\delta_D(x)\delta_D(y)}{(\delta_D(x) \vee \delta_D(y) \vee |x-y|)^2} &\le \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) \\ &\le \frac{9}{4} \frac{\delta_D(x)\delta_D(y)}{(\delta_D(x) \vee \delta_D(y) \vee |x-y|)^2}. \end{aligned} \tag{3.10}$$

(See [3] for the proof.)

Proof of Theorem 1.3. Fix a half-space-like $C^{1,1}$ open set D with $C^{1,1}$ characteristics (R, Λ_0). Without loss of generality, we assume $M = 1$, $\delta_D(x) \le \delta_D(y)$ and that $H_{1/4} \subset D \subset H_{-1/4}$, where $H_a := \{y = (\widehat{y}, y_d) \in \mathbb{R}^d : y_d > a\}$. In this proof, the notation $f \asymp g$ means that there are positive constants c_1 and c_2 depending only on D and α so that $c_1 g(x) \le f(x) \le c_2 g(x)$ in the common domain of definition for f and g.

We first deal with the case $d \ge 2$. If D is a general $C^{1,1}$ open set, by [7, (4.3), (4.4), (4.6), (4.7)] and our Theorem 1.1(i), we have for $|x-y| \le 3m^{-1/\alpha}$,

$$\int_0^1 p^m_D(t, x, y)dt \asymp \frac{1}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|^\alpha}\right). \tag{3.11}$$

Now we assume that D is a half-space-like $C^{1,1}$ open set in $\mathbb{R}^d$. Observe that if $\delta_D(x) \ge 1$, then

$$\frac{1}{2}\delta_D(x) \le \delta_D(x) - \frac{1}{2} \le \delta_{H_{1/4}}(x) \le \delta_D(x) \le \delta_{H_{-1/4}}(x) \le \delta_D(x) + \frac{1}{2} \le \frac{3}{2}\delta_D(x). \tag{3.12}$$

If $1 \le \delta_D(x) \le \delta_D(y)$, by (3.12) and Theorem 3.1, we know that there exists $c_3 > 1$ such that for all $m > 0$ and $d \ge 2$,

$$c_3^{-1} \widetilde{V}^{\alpha,m}_D(x, y) \le G^m_{H_{1/4}}(x, y) \le G^m_D(x, y) \le G^m_{H_{-1/4}}(x, y) \le c_3 \widetilde{V}^{\alpha,m}_D(x, y). \tag{3.13}$$

Now we consider the case $\delta_D(x) < 1$. In the remainder of this proof, for each $x \in D$, we define

$$x_0 := (\widehat{x}, x_d + 1). \tag{3.14}$$

(i) First we assume that $|x-y| > 3m^{-1/\alpha}$. In this case, we have $|x_0-y| > 2m^{-1/\alpha}$. If $R/3 \le \delta_D(x) < 1$, by the uniform boundary Harnack principle (Theorem 2.6),

$$G_D^m(x,y) \asymp G_D^m(x_0,y) \asymp \delta_D(x)^{\alpha/2} G_D^m(x_0,y). \tag{3.15}$$

If $\delta_D(x) < R/3$, choose a $Q_x \in \partial D$ such that $\delta_D(x) = |x-Q_x|$. Note that $x_0 \in B(Q_x,2)$. It follows from [28, Lemma 2.2] that one can choose a constant $c_4 = c_4(D)$ and a bounded $C^{1,1}$ open set U, whose $C^{1,1}$-characteristics depends on D but is independent of x, such that $B(Q_x, \frac{10}{4}R) \cap D \subset U \subset B(Q_x, \frac{11}{4}R) \cap D$ and $(U \cap \{\delta_U(z) > c_4\}) \setminus B(Q_x, \frac{9}{4}R)$ is nonempty. Note that $\delta_U(x) = \delta_D(x)$. Choose an $x_1 \in (U \cap \{\delta_U(z) > c_4\}) \setminus B(Q_x, \frac{9}{4}R)$. By the uniform boundary Harnack principle (Theorem 2.6), the uniform Harnack inequality (Theorem 2.3) and Theorem 1.2,

$$G_D^m(x,y) \asymp \frac{G_U^m(x,x_1)}{G_U^m(x_0,x_1)} G_D^m(x_0,y) \asymp \delta_U(x)^{\alpha/2} G_D^m(x_0,y) = \delta_D(x)^{\alpha/2} G_D^m(x_0,y). \tag{3.16}$$

If $\delta_D(x) < 1 \le \delta_D(y)$, then

$$\delta_{H_{1/4}}(y) \le \delta_D(y) \le \delta_D(x) + |x-y| \le 1 + |x-y| \le \frac{4}{3}|x-y|. \tag{3.17}$$

By Theorem 3.1, (3.12), (3.15) and (3.16), we have for $d \ge 3$,

$$\begin{aligned}
G_D^m(x,y) &\asymp \delta_D(x)^{\alpha/2} G_D^m(x_0,y) \le \delta_D(x)^{\alpha/2} G_{H_{-1/4}}^m(x_0,y) \\
&\asymp \delta_D(x)^{\alpha/2} m^{(2-\alpha)/\alpha} \left(1 \wedge \frac{(1+m^{-(2-\alpha)/(2\alpha)})(\delta_{H_{-1/4}}(y) + m^{-(2-\alpha)/(2\alpha)}\delta_{H_{-1/4}}(y)^{\alpha/2})}{|x_0-y|^2}\right) \\
&\quad \times \frac{1}{|x_0-y|^{d-2}} \\
&\asymp m^{(2-\alpha)/\alpha} \left(\delta_D(x)^{\alpha/2} \wedge \frac{(\delta_D(x)^{\alpha/2} + m^{-(2-\alpha)/(2\alpha)}\delta_D(x)^{\alpha/2})(\delta_D(y) + m^{-(2-\alpha)/(2\alpha)}\delta_D(y)^{\alpha/2})}{|x-y|^2}\right) \\
&\quad \times \frac{1}{|x-y|^{d-2}} \\
&\le m^{(2-\alpha)/\alpha} \left(1 \wedge \frac{(\delta_D(x) + m^{-(2-\alpha)/(2\alpha)}\delta_D(x)^{\alpha/2})(\delta_D(y) + m^{-(2-\alpha)/(2\alpha)}\delta_D(y)^{\alpha/2})}{|x-y|^2}\right) \\
&\quad \times \frac{1}{|x-y|^{d-2}} \\
&= \widetilde{V}^{\alpha,m}(x,y)
\end{aligned}$$

and

$$\begin{aligned}
G_D^m(x,y) &\asymp \delta_D(x)^{\alpha/2} G_D^m(x_0,y) \ge \delta_D(x)^{\alpha/2} G_{H_{1/4}}^m(x_0,y) \\
&\asymp \delta_D(x)^{\alpha/2} m^{(2-\alpha)/\alpha} \frac{\left(1+m^{-(2-\alpha)/(2\alpha)}\right)\left(\delta_{H_{1/4}}(y) + m^{-(2-\alpha)/(2\alpha)}\delta_{H_{1/4}}(y)^{\alpha/2}\right)}{|x_0-y|^2} \\
&\quad \times \frac{1}{|x_0-y|^{d-2}} \\
&\asymp m^{(2-\alpha)/\alpha} \frac{\left(\delta_D(x) + m^{-(2-\alpha)/(2\alpha)}\delta_D(x)^{\alpha/2}\right)\left(\delta_D(y) + m^{-(2-\alpha)/(2\alpha)}\delta_D(y)^{\alpha/2}\right)}{|x-y|^2}
\end{aligned}$$

$$\times \frac{1}{|x-y|^{d-2}}$$
$$\geq \widetilde{V}^{\alpha,m}(x,y),$$

where in the second line of the display above we used (3.17).

Similarly, when $\delta_D(x) < 1 \leq \delta_D(y)$ and $d = 2$, we get

$$G_D^m(x,y) \asymp m^{(2-\alpha)/\alpha} \times \log\left(1 + \frac{(\delta_D(x) + m^{-(2-\alpha)/(2\alpha)}\delta_D(x)^{\alpha/2})(\delta_D(y) + m^{-(2-\alpha)/(2\alpha)}\delta_D(y)^{\alpha/2})}{|x-y|^2}\right).$$

Now we suppose that $\delta_D(x) \leq \delta_D(y) < 1$. In this case we have $|x_0 - y_0| = |x-y| > 3m^{-1/\alpha}$. Repeating the argument before (3.16) with y instead of x, we get

$$G_D^m(x,y) \asymp \delta_D(x)^{\alpha/2} G_D^m(x_0,y) \asymp \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} G_D^m(x_0,y_0). \tag{3.18}$$

Thus by (3.12), (3.18) and Theorem 3.1, we have for $d \geq 3$,

$$\begin{aligned} G_D^m(x,y) &\asymp \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} G_D^m(x_0,y_0) \leq \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} G_{H_{-1/4}}^m(x_0,y_0) \\ &\asymp \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} m^{(2-\alpha)/\alpha} \frac{\left(1 + m^{-(2-\alpha)/(2\alpha)}\right)\left(1 + m^{-(2-\alpha)/(2\alpha)}\right)}{|x_0-y_0|^2} \frac{1}{|x_0-y_0|^{d-2}} \\ &\asymp m^{(2-\alpha)/\alpha} \frac{\left(\delta_D(x) + m^{-(2-\alpha)/(2\alpha)}\delta_D(x)^{\alpha/2}\right)\left(\delta_D(y) + m^{-(2-\alpha)/(2\alpha)}\delta_D(y)^{\alpha/2}\right)}{|x-y|^2} \\ &\quad \times \frac{1}{|x-y|^{d-2}} \\ &\asymp \widetilde{V}^{\alpha,m}(x,y) \end{aligned}$$

and

$$\begin{aligned} G_D^m(x,y) &\asymp \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} G_D^m(x_0,y_0) \geq \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} G_{H_{1/4}}^m(x_0,y_0) \\ &\asymp \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} m^{(2-\alpha)/\alpha} \frac{\left(1 + m^{-(2-\alpha)/(2\alpha)}\right)\left(1 + m^{-(2-\alpha)/(2\alpha)}\right)}{|x_0-y_0|^2} \\ &\quad \times \frac{1}{|x_0-y_0|^{d-2}} \\ &\asymp m^{(2-\alpha)/\alpha} \frac{\left(\delta_D(x) + m^{-(2-\alpha)/(2\alpha)}\delta_D(x)^{\alpha/2}\right)\left(\delta_D(y) + m^{-(2-\alpha)/(2\alpha)}\delta_D(y)^{\alpha/2}\right)}{|x-y|^2} \\ &\quad \times \frac{1}{|x-y|^{d-2}} \\ &\asymp \widetilde{V}^{\alpha,m}(x,y). \end{aligned}$$

Similarly, when $\delta_D(x) \leq \delta_D(y) < 1$, $|x-y| > 3m^{-1/\alpha}$ and $d = 2$, we have

$$G_D^m(x,y) \asymp m^{(2-\alpha)/\alpha} \times \log\left(1 + \frac{(\delta_D(x) + m^{-(2-\alpha)/(2\alpha)}\delta_D(x)^{\alpha/2})(\delta_D(y) + m^{-(2-\alpha)/(2\alpha)}\delta_D(y)^{\alpha/2})}{|x-y|^2}\right).$$

(ii) Now assume that $|x-y| \le 3m^{-1/\alpha}$. By Theorem 1.1, we have

$$
\begin{aligned}
&\int_1^\infty p_D^m(t,x,y)dt \\
&\le c_5(1\wedge\delta_D(x))^{\alpha/2}(1\wedge\delta_D(y))^{\alpha/2}\int_{D\times D}(1\wedge\delta_D(z))^{\alpha/2}\left(1\wedge\frac{\phi(m^{1/\alpha}|x-z|/C_2)}{|x-z|^{d+\alpha}}\right)\\
&\quad\times G_D^m(z,w)(1\wedge\delta_D(w))^{\alpha/2}\left(1\wedge\frac{\phi(m^{1/\alpha}|w-y|/C_2)}{|w-y|^{d+\alpha}}\right)dzdw\\
&\le c_5(1\wedge\delta_D(x))^{\alpha/2}(1\wedge\delta_D(y))^{\alpha/2}\\
&\quad\times\int_{H_{-1}\times H_{-1}}(1\wedge\delta_{H_{-1}}(z))^{\alpha/2}\left(1\wedge\frac{\phi(m^{1/\alpha}|x-z|/C_2)}{|x-z|^{d+\alpha}}\right)\\
&\quad\times G_{H_{-1}}^m(z,w)\,(1\wedge\delta_{H_{-1}}(w))^{\alpha/2}\left(1\wedge\frac{\phi(m^{1/\alpha}|w-y|/C_2)}{|w-y|^{d+\alpha}}\right)dzdw. \qquad (3.19)
\end{aligned}
$$

Here we take $C_2>0$ to be the constant in Theorem 1.1(i) for both our D and the half spaces. Let $b := C_2^2$ and set $\widetilde{x} := x/b$, $\widetilde{y} = y/b$. Using the change of variables $\widetilde{z} = z/b$ and $\widetilde{w} = w/b$, by Theorem 1.1, we have that (3.19) is less than or equal to $(1\wedge\delta_D(x))^{\alpha/2}\,(1\wedge\delta_D(y))^{\alpha/2}$ times

$$
\begin{aligned}
&c_6\int_{H_{-1/b}\times H_{-1/b}}\left(1\wedge\delta_{H_{-1/b}}(\widetilde{x})\right)^{\alpha/2}\left(1\wedge\delta_{H_{-1/b}}(\widetilde{z})\right)^{\alpha/2}\left(1\wedge\frac{\phi(m^{1/\alpha}C_2|\widetilde{x}-\widetilde{z}|)}{|\widetilde{x}-\widetilde{z}|^{d+\alpha}}\right)\\
&\quad\times G_{H_{-1}}^m(z,w)\left(1\wedge\delta_{H_{-1/b}}(\widetilde{y})\right)^{\alpha/2}\\
&\quad\times\left(1\wedge\delta_{H_{-1/b}}(\widetilde{w})\right)^{\alpha/2}\left(1\wedge\frac{\phi(m^{1/\alpha})C_2|\widetilde{w}-\widetilde{y}|}{|\widetilde{w}-y|^{d+\alpha}}\right)dzdw\\
&\le c_7\int_{H_{-1/b}\times H_{-1/b}}p_{H_{-1/b}}^m(1,\widetilde{x},\widetilde{z})G_{H_{-1}}^m(b\widetilde{z},b\widetilde{w})p_{H_{-1/b}}^m(1,\widetilde{y},\widetilde{w})dzdw\\
&\le c_8\int_{H_{-1/b}\times H_{-1/b}}p_{H_{-1/b}}^m(1,\widetilde{x},\widetilde{z})G_{H_{-1/b}}^m(\widetilde{z},\widetilde{w})p_{H_{-1/b}}^m(1,\widetilde{y},\widetilde{w})dzdw\\
&= c_8\int_0^\infty\int_{H_{-1/b}\times H_{-1/b}}p_{H_{-1/b}}^m(1,\widetilde{x},\widetilde{z})p_{H_{-1/b}}^m(t,\widetilde{z},\widetilde{w})p_{H_{-1/b}}^m(1,\widetilde{y},\widetilde{w})dzdwdt\\
&= c_8\int_0^\infty p_{H_{-1/b}}^m(t+2,\widetilde{x},\widetilde{y})dt, \qquad (3.20)
\end{aligned}
$$

where in the second inequality above we used (3.9) and the scaling property (2.7). Thus

$$
\int_1^\infty p_D^m(t,x,y)dt \le c_9(1\wedge\delta_D(x))^{\alpha/2}(1\wedge\delta_D(y))^{\alpha/2}G_{H_{-1/b}}^m(\widetilde{x},\widetilde{y}). \qquad (3.21)
$$

Now by Theorem 3.1, for $d\ge 3$,

$$
\begin{aligned}
&\int_1^\infty p_D^m(t,x,y)dt\\
&\le c_{10}(1\wedge\delta_D(x))^{\alpha/2}(1\wedge\delta_D(y))^{\alpha/2}\left(1\wedge\frac{\delta_{H_{-1/b}}(\widetilde{x})\delta_{H_{-1/b}}(\widetilde{y})}{|\widetilde{x}-\widetilde{y}|^2}\right)^{\alpha/2}|\widetilde{x}-\widetilde{y}|^{\alpha-d}
\end{aligned}
$$

$$\le c_{11}(1 \wedge \delta_D(x))^{\alpha/2}(1 \wedge \delta_D(y))^{\alpha/2}\left(1 \wedge \frac{(\delta_D(x)+1)(\delta_D(y)+1)}{|x-y|^2}\right)^{\alpha/2}|x-y|^{\alpha-d}$$

$$\le c_{12}\left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2}|x-y|^{\alpha-d}. \tag{3.22}$$

Thus when $d \ge 3$, by (3.11) and (3.22), we have

$$c_{13}\left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2}|x-y|^{\alpha-d} \le G_D^m(x,y)$$
$$= \int_0^1 p_D^m(t,x,y)dt + \int_1^\infty p_D^m(t,x,y)dt$$
$$\le c_{14}\left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2}|x-y|^{\alpha-d}. \tag{3.23}$$

We conclude from this that $G_D^m(x,y) \asymp \widetilde{V}_D^{\alpha,m}(x,y)$. This completes the proof for the case $d \ge 3$.

Now we deal with the case $d = 2$. Note that $m^{(2-\alpha)/\alpha}\log(1 \vee m^{1/\alpha}(\delta_D(x) \wedge \delta_D(y))) = 0$ as $\delta_D(x) < 1$ and M is assumed to be 1. By Theorem 3.1 and (3.21), when $\delta_D(x) < 1$, we have

$$\int_1^\infty p_D^m(t,x,y)dt$$
$$\le c_{15}(1 \wedge \delta_D(x))^{\alpha/2}(1 \wedge \delta_D(y))^{\alpha/2}\left(1 \wedge \frac{(\delta_D(x)+1)(\delta_D(y)+1)}{|x-y|^2}\right)^{\alpha/2}|x-y|^{\alpha-2}$$
$$+ c_{15}(1 \wedge \delta_D(x))^{\alpha/2}(1 \wedge \delta_D(y))^{\alpha/2}m^{(2-\alpha)/\alpha}$$
$$\times \log(1 \vee m^{1/\alpha})((\delta_D(x)+2/b) \wedge (\delta_D(y)+2/b))$$
$$\le c_{15}(1 \wedge \delta_D(x))^{\alpha/2}(1 \wedge \delta_D(y))^{\alpha/2}$$
$$\times \left(\left(1 \wedge \frac{(\delta_D(x)+1)(\delta_D(y)+1)}{|x-y|^2}\right)^{\alpha/2}|x-y|^{\alpha-2} + \log(1 \vee m^{1/\alpha}(1+2/b))\right). \tag{3.24}$$

Note that the second term in (3.24) is non-zero if and only if $m > (1+2/b)^{-\alpha}$. Thus in that case, we have that $|x-y| < 3(1+2/b)$ and so

$$\log(1 \vee m^{1/\alpha}(1+2/b)) \le c_{16} \le c_{17}\left(1 \wedge \frac{(\delta_D(x)+1)(\delta_D(y)+1)}{|x-y|^2}\right)^{\alpha/2}|x-y|^{\alpha-2}. \tag{3.25}$$

Therefore, using (3.11), (3.24) and (3.25), (3.23) is true for $d = 2$ when $\delta_D(x) < 1$, and we conclude from this and (3.13) that $G_D^m(x,y) \asymp \widetilde{V}_D^{\alpha,m}(x,y)$ for $d = 2$.

Now we deal with the case $d = 1$. By following the arguments in parts (ii) and (iii) of the proof of [7, Corollary 1.2], our Theorem 1.1(i) gives that for $|x-y| \le 3m^{-1/\alpha}$,

$$\int_0^1 p_D^m(t,x,y)dt$$

$$\asymp \begin{cases} \dfrac{1}{|x-y|^{1-\alpha}}\left(1\wedge \dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha}}\right) & \text{when } d=1>\alpha, \\ \left(\delta_D(x)\delta_D(y)\right)^{(\alpha-1)/2}\wedge \dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|} & \text{when } d=1<\alpha, \\ \log\left(1+\dfrac{\delta_D(x)^{1/2}\delta_D(y)^{1/2}}{|x-y|}\right) & \text{when } d=1=\alpha. \end{cases} \tag{3.26}$$

Clearly (3.12) holds for $d=1$ as well.

If $1\le\delta_D(x)\le\delta_D(y)$, by (3.12) and Theorem 3.1, we know that there exists $c_{18}>1$ such that for all $m>0$,

$$c_{18}^{-1}\widetilde{V}_D^{\alpha,m}(x,y)\le G^m_{H_{1/4}}(x,y)\le G^m_D(x,y)\le G^m_{H_{-1/4}}(x,y)\le c_{18}\widetilde{V}_D^{\alpha,m}(x,y).$$

Thus in the remainder of the proof we assume that $\delta_D(x)<1$.

(i) First assume that $|x-y|>3m^{-1/\alpha}$. Recall that we have assumed that $M=1$, $\delta_D(y)\ge\delta_D(x)$ and that $H_{1/4}\subset D\subset H_{-1/4}$. It follows in this one-dimensional case that $\delta_D(y)\ge|x-y|-\delta_D(x)-1/2\ge 3/2$. Observe also that in view of (3.10), we have

$$\begin{aligned} \delta_D(x)\left(1\wedge\frac{\delta_D(y)}{|x-y|^2}\right) &\asymp \frac{\delta_D(x)\delta_D(y)}{(1\vee\delta_D(y)\vee|x-y|)^2} = \frac{\delta_D(x)\delta_D(y)}{(\delta_D(x)\vee\delta_D(y)\vee|x-y|)^2} \\ &\asymp \left(1\wedge\frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right). \end{aligned} \tag{3.27}$$

With (3.27) and Theorem 3.1, by the same argument as for the corresponding part above in the $d\ge2$ case, we can conclude that $G^m_D(x,y)\asymp\widetilde{V}^{\alpha,m}(x,y)$. We skip the details here.

(ii) Now assume that $|x-y|\le 3m^{-1/\alpha}$. By (3.21), we have

$$\int_1^\infty p^m_D(t,x,y)dt\le c_9(1\wedge\delta_D(x))^{\alpha/2}(1\wedge\delta_D(y))^{\alpha/2}G^m_{H_{-1/b}}(\widetilde{x},\widetilde{y}) \tag{3.28}$$

where $b:=C_2^2$ and $\widetilde{x}:=x/b$, $\widetilde{y}=y/b$. By Theorem 3.1 and the fact

$$\begin{aligned} &\left(\delta_{H_{-1/b}}(\widetilde{x})\delta_{H_{-1/b}}(\widetilde{y})\right)^{(\alpha-1)/2}\wedge\frac{\delta_{H_{-1/b}}(\widetilde{x})^{\alpha/2}\delta_{H_{-1/b}}(\widetilde{y})^{\alpha/2}}{|\widetilde{x}-\widetilde{y}|} \\ &\asymp\frac{\delta_{H_{-1/b}}(\widetilde{x})^{\alpha/2}\delta_{H_{-1/b}}(\widetilde{y})^{\alpha/2}}{\delta_{H_{-1/b}}(\widetilde{x})\vee\delta_{H_{-1/b}}(\widetilde{y})\vee|\widetilde{x}-\widetilde{y}|}, \end{aligned}$$

we have

$$G^m_{H_{-1/b}}(\widetilde{x},\widetilde{y})\asymp\begin{cases} \dfrac{\delta_{H_{-1/b}}(\widetilde{x})^{\alpha/2}\delta_{H_{-1/b}}(\widetilde{y})^{\alpha/2}}{\delta_{H_{-1/b}}(\widetilde{x})\vee\delta_{H_{-1/b}}(\widetilde{y})\vee|\widetilde{x}-\widetilde{y}|} & \text{when } \alpha>1, \\ \log\left(1+\dfrac{\delta_{H_{-1/b}}(\widetilde{x})^{1/2}\delta_{H_{-1/b}}(\widetilde{y})^{1/2}}{|\widetilde{x}-\widetilde{y}|}\right) & \text{when } \alpha=1 \\ \left(1\wedge\dfrac{\delta_{H_{-1/b}}(\widetilde{x})\delta_{H_{-1/b}}(\widetilde{y})}{|\widetilde{x}-\widetilde{y}|^2}\right)^{\alpha/2}|\widetilde{x}-\widetilde{y}|^{\alpha-1} & \text{when } \alpha<1. \end{cases} \tag{3.29}$$

Thus

$$G^m_{H_{-1/b}}(\widetilde{x},\widetilde{y}) \le c_{19} \begin{cases} \dfrac{(\delta_D(x)+1)^{\alpha/2}(\delta_D(y)+1)^{\alpha/2}}{\delta_D(x)\vee\delta_D(y)\vee|x-y|} & \text{when } \alpha>1, \\ \log\left(1+\dfrac{(\delta_D(x)+1)^{1/2}(\delta_D(y)+1)^{1/2}}{|x-y|}\right) & \text{when } \alpha=1 \\ \left(1\wedge\dfrac{(\delta_D(x)+1)(\delta_D(y)+1)}{|x-y|^2}\right)^{\alpha/2}|x-y|^{\alpha-1} & \text{when } \alpha<1. \end{cases}$$

Note that, for $0<b<1$, the function $f(z)=\log(1+bz)-b\log(1+z)>0$ on $(0,\infty)$ since $f(0)=0$ and $f'(z)=b\left((1+bz)^{-1}-(1+z)^{-1}\right)>0$ for $z>0$. Thus

$$\begin{aligned}&(1\wedge\delta_D(x))^{1/2}(1\wedge\delta_D(y))^{1/2}\log\left(1+\frac{(\delta_D(x)+1)^{1/2}(\delta_D(y)+1)^{1/2}}{|x-y|}\right)\\ &\quad\le\log\left(1+\frac{(1\wedge\delta_D(x))^{1/2}(1\wedge\delta_D(y))^{1/2}(\delta_D(x)+1)^{1/2}(\delta_D(y)+1)^{1/2}}{|x-y|}\right).\end{aligned} \tag{3.30}$$

Therefore by (3.28),

$$\begin{aligned}&\int_1^\infty p^m_D(t,x,y)dt\\ &\le c_{20}\begin{cases}(1\wedge\delta_D(x))^{\alpha/2}(1\wedge\delta_D(y))^{\alpha/2}\dfrac{(\delta_D(x)+1)^{\alpha/2}(\delta_D(y)+1)^{\alpha/2}}{\delta_D(x)\vee\delta_D(y)\vee|x-y|}\\ \quad\text{when } \alpha>1,\\ \log\left(1+\dfrac{(1\wedge\delta_D(x))^{1/2}(1\wedge\delta_D(y))^{1/2}(\delta_D(x)+1)^{1/2}(\delta_D(y)+1)^{1/2}}{|x-y|}\right)\\ \quad\text{when } \alpha=1,\\ (1\wedge\delta_D(x))^{\alpha/2}(1\wedge\delta_D(y))^{\alpha/2}\left(1\wedge\dfrac{(\delta_D(x)+1)(\delta_D(y)+1)}{|x-y|^2}\right)^{\alpha/2}|x-y|^{\alpha-1}\\ \quad\text{when } \alpha<1\end{cases}\\ &\le c_{21}\begin{cases}\dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{\delta_D(x)\vee\delta_D(y)\vee|x-y|} & \text{when } \alpha>1,\\ \log\left(1+\dfrac{\delta_D(x)^{1/2}\delta_D(y)^{1/2}}{|x-y|}\right) & \text{when } \alpha=1,\\ \left(1\wedge\dfrac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2}|x-y|^{\alpha-1} & \text{when } \alpha<1.\end{cases}\end{aligned}$$

Combining this with (3.26), we obtain

$$\begin{aligned}\widetilde{V}^{\alpha,m}_D(x,y)&\asymp\int_0^1 p^m_D(t,x,y)dt\le G^m_D(x,y)\\ &=\int_0^1 p^m_D(t,x,y)dt+\int_1^\infty p^m_D(t,x,y)dt\le c_{22}\widetilde{V}^{\alpha,m}_D(x,y).\end{aligned}$$

This completes the proof for the case $d=1$. □

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